Strings on Plane Waves and $AdS \times S$

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Abstract

We consider the RNS superstrings in $AdS_3 \times S^3 \times \mathcal{M}$, where $\mathcal{M}$ may be $K3$ or T^4, based on $SL(2, R)$ and $SU(2)$ WZW models. We construct the physical states and calculate the spectrum. A subsector of this theory describes strings propagating in the six dimensional plane wave obtained by the Penrose limit of $AdS_3 \times S^3 \times \mathcal{M}$. We reproduce the plane wave spectrum by taking J and the radius to infinity. We show that the plane wave spectrum actually coincides with the large J spectrum at fixed radius, i.e. in $AdS_3 \times S^3$. Relation to some recent topics of interest such as the Frolov-Tseytlin string and strings with critical tension or in zero radius AdS are discussed.

Contents

1. Introduction

Understanding the duality between gauge theories and string theories has been a topic of great interest, ever since [1]. The AdS/CFT correspondence [2,3,4,5] has provided a wealth of examples in which string theories compactified on spacetimes that include an AdS factor are dual to field theories on the boundary of the AdS. However, progress has been limited due to the fact these backgrounds involve R-R fields, and solving for the string spectrum in such spacetimes remains an outstanding problem. Lacking an exact string theory, the low-energy supergravity approximation has been used for most part.

Recently an exciting set of new dualities was obtained by Berenstein, Maldacena, and Nastase [6]. By taking a limit of $AdS_5 \times S^5$ in which the geometry becomes that of a plane wave, one obtains a background that allows for exact string quantization, in Green-Schwarz formalism [7]. The limiting procedure involves taking the radius of $AdS_5 \times S^5$ to infinity and is an example of Penrose's limit [8]. At the same time, on the CFT side one focuses on those states with large conformal weight and R-charge: $\Delta, J \to \infty$ as R^2, but with finite $\Delta - J$. In this way each AdS/CFT duality gives rise to a plane wave/CFT duality, in which one may go beyond the supergravity approximation. Specifically, BMN was able to reproduce, from the CFT point of view, some of the *stringy* excitations in the plane wave. This represents remarkable progress towards establishing the correspondence between a fully string theoretic description of gravity on AdS and the CFT on the boundary.

Furthermore, it has been shown that some physical quantities of interest may be computed perturbatively on both sides of the BMN correspondence [9,10,11,12]. This differs from AdS/CFT, in which the duality relates the weak coupling physics on one side to the strong coupling physics on the other. This development has led to an intense level of activity[1] which has resulted in significant understanding of both gauge theory and string field theory.

For these reasons string theory on plane waves that arise as Penrose limits of $AdS \times S$ has emerged as a topic of great importance. However, the GS formulation of superstrings is technically cumbersome and much insight would be gained from an example of an exact CFT description of string propagation on a plane wave. Happily, such an example exists: the plane wave obtained via the Penrose limit of $AdS_3 \times S^3$ with a purely NS-NS field strength. One of the goals of this paper is to study superstrings in this background using CFT techniques. For earlier work on the $AdS_3 \times S^3$ plane wave, see [26,27,28,29,30,31].

Actually, the $AdS_3 \times S^3$ plane wave with NS background is special for another reason– string theory is solvable even before the Penrose limit is taken! The CFT on the string worldsheet is given by the $SL(2,R)$ and $SU(2)$ WZW models with the level of the current algebras determined by the radius of $AdS_3 \times S^3$. With the recent understanding of the $SL(2,R)$ WZW model [32,33,34], the string spectrum in AdS_3 has been found and correlation functions have been calculated. The solvability of string theory on $AdS_3 \times S^3$ allows us to view string theory on the six dimensional plane wave as one of its subsectors. This is similar in spirit to how the $\mathcal{N} = 4$ SYM theory is studied in the ten dimensional BMN duality, in that one does not in anyway change the theory while trying to study the correspondence. Rather, one restricts focus onto a particular subclass of operators, such as the (nearly) chiral operators, for which it is possible to say something about the dual objects in the string side.

In the same manner, we begin with the full string spectrum on $AdS_3 \times S^3 \times \mathcal{M}$ at arbitrary values of the level k and angular momentum J on S^3. As we take $k, J \to \infty$, we can "see" how the Hilbert space breaks apart, and a subspace arising in this limit corresponds to the plane wave Hilbert space. The spectral flow symmetry of the $SL(2,R)$ WZW model found in [32] once again will play a key role in this discussion[2].

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[1] See, for example, [13,14,15,16,17,19,20,21], and [22,23,24,25] for reviews.

[2] Previous work on the plane wave limit of $AdS_3 \times S^3$ either did not address the issue of spectral flow, or discussed it as a symmetry of the WZW model based on the extended Heisenberg group, i.e. after the Penrose limit was taken.

Moreover, since our treatment is fully string theoretic from the start, or in other words valid for arbitrary values of the radius, we can attempt to address the following important question: What can string theory on plane waves tell us about string theory on $AdS \times S$? Even though it is believed that the former represents a great simplification of the latter (the plane wave is, after all, just the first term in an R^{-2} expansion of $AdS \times S$), we find some strong evidence that in fact some aspects of string theory in the plane wave could be trusted away from the strict $R^2 \to \infty$ limit. Specifically, we will show that the large J spectrum of strings on $AdS_3 \times S^3$ with NS background at finite R^2 coincides with the plane wave spectrum, found in [6,26,27,28]. This is rather surprising since the spacetime geometry in each case is drastically different. Our result provides an explicit and compelling evidence in support of some of the recent ideas [35,36,37,38,39] about extrapolating the semiclassical relationship between energy and spins in $AdS_5 \times S^5$ down to the stringy regime.

This paper is organized as follows. We begin by briefly reviewing in section 2 the relevant aspects of string theory on $AdS_3 \times S^3 \times \mathcal{M}$, where $\mathcal{M}$ may be $K3$ or T^4. We first review the bosonic case, then turn to the case of superstrings which will be the subject of our focus. In section 3, we describe the Penrose limit which takes $AdS_3 \times S^3$ to the plane wave. In section 4, we study the semi-classical limit of strings that will be relevant in the plane wave limit. We do this by computing the Nambu action of a string near the origin of AdS_3 and moving with high angular momentum on a great circle of S^3. This is the six dimensional analog of the particle trajectory used by BMN to obtain the ten dimensional plane wave from $AdS_5 \times S^5$ [6]. The resulting Nambu action displays the same behavior as what was shown in [32]. Namely, new representations that do not obey the usual highest weight conditions appear. These representations are obtained from the usual representations by spectral flow, and it is shown that the amount of spectral flow depends on the ratio of the angular momentum to R^2. Armed with this knowledge, in section 5 we obtain the exact string spectrum on $AdS_3 \times S^3$, valid for arbitrary values of R^2 and J. The plane wave spectrum is reproduced by taking $R^2, J \to \infty$ and expanding to leading order. In section 6 we discuss the decoupling of the Hilbert space in the Penrose limit. In section 7 we discuss what happens when the radius of $AdS_3 \times S^3$ is finite. Section 8 contains a summary and discussion. In Appendix A we show how the spectral flow number violation rule found in [34] can be understood in terms of angular momentum conservation in the plane wave.

2. String theory on $AdS_3 \times S^3 \times \mathcal{M}$

We begin with a review of the bosonic string theory in AdS_3 as discussed in [32][3]. By considering new representations of $\widehat{SL}(2, R)$ algebra that are obtained by the spectral flow operation, authors of [32] showed that the long-standing problem of an apparent upper limit on the energy of string states [40] was eliminated, and that the spectrum consisted of short strings and long strings [41,42]. This proposal was verified through a one-loop string calculation in [33], and correlation functions were computed in [34].

2.1. $SL(2, R)$ WZW model

Since $AdS_3 \cong SL(2, R)$[4], the CFT on the string worldsheet is given by the $SL(2, R)$ WZW model. The action is

$$S = \frac{k}{8\pi\alpha'} \int d^2z \, \mathrm{Tr}(g^{-1}\partial g g^{-1}\bar{\partial}g) + k\Gamma_{WZ} \,, \tag{2.1}$$

where g is an element of $SL(2, R)$. The parametrization of g is given by

$$g = e^{i(t+\phi)\sigma^2/2}e^{\rho\sigma^3}e^{i(t-\phi)\sigma^2/2} \,, \tag{2.2}$$

which corresponds to parametrization of AdS_3 in the global coordinates

$$ds^2 = -\cosh^2\rho \, dt^2 + d\rho^2 + \sinh^2\rho \, d\phi^2 \,. \tag{2.3}$$

The level k need not be quantized since H^3 is trivial for $SL(2, R)$. However, when we consider (super)string theory on $AdS_3 \times S^3$, the level of the two WZW models will turn out to be equal, and owing to the quantization of the level in the $SU(2)$ model k is restricted to be an integer.

This model has $\widehat{SL}(2, R)_L \times \widehat{SL}(2, R)_R$ symmetry, generated by two commuting copies of the current algebra

$$\begin{aligned}
\left[K_m^+, K_n^-\right] &= -2K_{m+n}^3 + km\delta_{m+n} \\
\left[K_m^3, K_n^\pm\right] &= \pm K_{m+n}^\pm \\
\left[K_m^3, K_n^3\right] &= -\frac{k}{2}m\delta_{m+n} \,.
\end{aligned} \tag{2.4}$$

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[3] The list of papers on AdS_3 preceding [32] is too long to be referenced here. We direct the reader to [32] for an extensive bibliography.

[4] We always consider the universal cover of $SL(2, R)$, obtained by unwrapping the t coordinate.

The Virasoro generators are given by the Sugawara form (we only write the holomophic side from now on)

$$L_n = \frac{1}{k-2} \sum_{m=-\infty}^{\infty} : \eta_{ab} J_m^a J_{n-m}^b : , \tag{2.5}$$

where η_{ab} is the metric on $SL(2,R)$ with signature $(+,+,-)$. The generators (2.5) obey the Virasoro algebra with central charge

$$c = \frac{3k}{k-2} . \tag{2.6}$$

States of $SL(2,R)$ WZW model fall into representations of $\widehat{SL}(2,R)_L \times \widehat{SL}(2,R)_R$ current algebra. The question is which representations appear in the Hilbert space. Representations of current algebra can be constructed by considering representations of the global algebra, generated by the zero modes of the currents $K_0^{\pm,3}$, to be the primary states annhiliated by $K_{m>0}^{\pm,3}$. Then $K_{m<0}^{\pm,3}$ can be applied to these states, generating the representation of the current algebra. Hence, the first problem is to find the right representations of $SL(2,R)$. In [32], the following was proposed. The representations of $SL(2,R)$ that appear are $\mathcal{D}_\ell$ and $C_{\ell,\alpha}$, where $\mathcal{D}_\ell$ is the discrete lowest weight representation

$$\mathcal{D}_\ell = \{|\ell, n\rangle : n = \ell, \ell+1, \ell+2, \ldots\} , \tag{2.7}$$

with $K_0^- |\ell, \ell\rangle = 0$. The representation is labeled by the value of the quadratic Casimir

$$\left(\frac{1}{2}(K_0^+ K_0^- + K_0^- K_0^+) - (K_0^3)^2 \right) |\ell, n\rangle = -\ell(\ell-1)|\ell, n\rangle , \tag{2.8}$$

and n which is the eigenvalue of K_0^3, related to the spacetime energy by

$$E = K_0^3 + \bar{K}_0^3 . \tag{2.9}$$

$C_{\ell,\alpha}$ is the continuous representation

$$C_{\ell,\alpha} = \{|\ell, n, \alpha\rangle : n = \alpha, \alpha \pm 1, \alpha \pm 2, \ldots\} , \tag{2.10}$$

where without loss of generality α may be restricted to $0 \leq \alpha < 1$. Unitarity requires $\ell = 1/2 + is$ with s real. This gives for the quadratic Casimir

$$\left(\frac{1}{2}(K_0^+ K_0^- + K_0^- K_0^+) - (K_0^3)^2 \right) |\ell, n, \alpha\rangle = \left(\frac{1}{4} + s^2 \right) |\ell, n, \alpha\rangle . \tag{2.11}$$

Now starting with the above representations of $SL(2,R)$, representations of $\widehat{SL}(2,R)$ are generated by applying $K^a_{m<0}$. The resulting representations are denoted $\hat{\mathcal{D}}_\ell$ and $\hat{C}_{\ell,\alpha}$. However, these are not the only representations of $\widehat{SL}(2,R)$ that may appear. There are additional representations that are generated by the action of spectral flow [32]

$$
\begin{aligned}
K^3_m &\to \tilde{K}^3_m = K^3_m - \frac{k}{2}w\delta_{m,0} \\
K^+_m &\to \tilde{K}^+_m = K^+_{m+w} \\
K^-_m &\to \tilde{K}^-_m = K^-_{m-w} \; ,
\end{aligned}
\tag{2.12}
$$

and the resulting transformation on the Virasoro generators

$$
\tilde{L}_m = L_m + wK^3_m - \frac{k}{4}w^2\delta_{m,0} \; .
\tag{2.13}
$$

For each integer valued spectral flow, we generate the representations $\hat{\mathcal{D}}^w_\ell$ and $\hat{C}^w_{\ell,\alpha}$ from $\hat{\mathcal{D}}_\ell$ and $\hat{C}_{\ell,\alpha}$, respectively.

We are now ready to state the Hilbert space of $SL(2,R)$ WZW model. It is given by

$$
\mathcal{H}_{SL} = \oplus_{w=-\infty}^{\infty} \left[\left(\int_{\frac{1}{2}}^{\frac{k-1}{2}} d\ell \; \hat{\mathcal{D}}^w_\ell \otimes \hat{\mathcal{D}}^w_\ell \right) \oplus \left(\int_{\frac{1}{2}+i\mathbf{R}} d\ell \int_0^1 d\alpha \; \hat{C}^w_{\ell,\alpha} \otimes \hat{C}^w_{\ell,\alpha} \right) \right] \; .
\tag{2.14}
$$

The requirement $\frac{1}{2} < \ell < \frac{k-1}{2}$ in the case of the discrete representations is actually more restrictive than what is allowed by the no-ghost theorem of strings in AdS_3 [43], which states that $\ell < \frac{k}{2}$, but is needed in order for spectral flow symmetry to close upon the representations consistent with harmonic analysis. For the continuous representations $\ell = \frac{1}{2} + is$ with s a real number. In the context of string theory on AdS_3, s is interpreted as the momentum in the radial direction, as implied by (2.11). Finally, we note that the representations with negative spectral flow are the complex conjugates of the representations with positive spectral flow, and in particular generate the $\widehat{SL}(2,R)$ representations built from discrete highest weight representations of $SL(2,R)$.

It is clear that the $SL(2,R)$ WZW model contains states of negative norm since the metric on AdS_3 is of indefinite signature. However, the situation is no worse than that of strings in flat space. There, the field in the time direction X^0 has the wrong sign and states created by this field have negative norm. The situation is remedied by the fact that in covariant quatization (which we are using for AdS_3), Virasoro constraints remove these states, and the resulting physical string spectrum is free of ghosts. In [32], it was shown that the same is true for strings on AdS_3, extending the earlier results of [43]. In the following, we will consider a critical bosonic string theory on $AdS_3 \times \mathcal{X}$. We will impose the Virasoro constraints on the product Hilbert space of $SL(2,R)$ WZW model and the CFT describing string propagation on $\mathcal{X}$, and obtain the physical spectrum.

2.2. Bosonic strings on $AdS_3 \times \mathcal{X}$

We assume that the CFT on $\mathcal{X}$ is unitary, with central charge

$$c_\mathcal{X} = 26 - \frac{3k}{k-2} . \qquad (2.15)$$

The Virasoro operators are given by the sum of the Virasoro operators for each CFT, $L_m = L_m^{SL} + L_m^\mathcal{X}$. Consider a state in the discrete representation of $SL(2, R)$ WZW model, tensored with a state from $\mathcal{X}$ with conformal weight h. The combined state is labeled as

$$|\tilde{\ell}, \tilde{n}, \tilde{N}, w, h\rangle , \qquad (2.16)$$

where N is the grade[5] of the $\widehat{SL}(2, R)$ descendent. Let us denote this state by $|\Omega\rangle$. Taking into account the spectral flow relations (2.12) and (2.13), the Virasoro constraints are

$$(L_0 - 1)|\Omega\rangle = \left(-\frac{\tilde{\ell}(\tilde{\ell}-1)}{k-2} + \tilde{N} - w\tilde{n} - \frac{kw^2}{4} + h - 1 \right) |\Omega\rangle = 0$$
$$L_m|\Omega\rangle = (\tilde{L}_m^{SL} - w\tilde{K}_m^3 + L_m^\mathcal{X})|\Omega\rangle = 0 , \qquad m \geq 1 . \qquad (2.17)$$

For discrete representations, $\tilde{n} = \tilde{\ell} + q$, with q an integer. Using this relation with the first equation in (2.17), $\tilde{\ell}$ is determined to be

$$\tilde{\ell} = \frac{1}{2} - \frac{k-2}{2}w + \sqrt{\frac{1}{4} + (k-2)\left(N + h - \frac{1}{2}w(w+1) - 1 \right)} , \qquad (2.18)$$

where N is the grade as measured by L_0, related to $\tilde{N}$ by $N = \tilde{N} - wq$. We impose the level matching condition $L_0 = \bar{L}_0$, and find the spacetime energy (2.9)

$$E = 1 + 2w + q + \bar{q} + \sqrt{1 + 4(k-2)\left(N_w + h - \frac{1}{2}w(w+1) - 1 \right)} . \qquad (2.19)$$

Note that the energy is discrete, even though in the $SL(2, R)$ WZW model (2.14) the spectrum of ℓ was continuous. These states correspond to the short strings that are trapped inside AdS_3.

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[5] The word grade is used here instead of level, as the latter will be reserved for k that appears in (2.1).

For states coming from the continuous representations, we can proceed in a similar manner to obtain their spectrum. The difference in this case is that $\tilde{\ell}$ and $\tilde{n}$ are not related. The result is

$$E = \frac{kw}{2} + \frac{1}{w}\left(\frac{2s^2 + \frac{1}{2}}{k-2} + \tilde{N} + \tilde{\bar{N}} + h + \bar{h} - 2\right) . \qquad (2.20)$$

Note that this time the grade is measured by $\tilde{L}_0$. The spectrum is continous and s represents the momentum of the string in the radial direction of AdS_3. These are the long strings [42,41] that can approach arbitrarily close to the boundary. Note that the continuous representations with $w = 0$ correspond to a tachyon in the spacetime. When we consider the supersymmetric case, such states will be projected out of the spectrum.

In this paper we we will be interested in the case where the internal space $\mathcal{X}$ contains S^3. For this we will need the $SU(2)$ WZW model, which we turn to next. We will briefly describe the Hilbert space of $SU(2)$ WZW model, in order to introduce notation and also because as we will see later, the analog of spectral flow (2.12), (2.13) in the $SU(2)$ WZW model will prove to be an useful tool in studying superstrings in the plane wave.

2.3. SU(2) WZW model

String theory on S^3 is described by the $SU(2)$ WZW model, and its Hilbert space can be constructed in a manner similar to what we described above for $SL(2,R)$. Again, we will restrict our attention to the holomorphic sector.

The action takes the same form as (2.1), but now with g labelling an element of $SU(2)$. The parametrization of the $SU(2)$ group manifold is very similar to what was used for $SL(2,R)$, eqn. (2.2). The metric on S^3 reads

$$ds^2 = \cos^2\theta d\psi^2 + d\theta^2 + \sin^2\theta d\varphi^2 . \qquad (2.21)$$

The symmetry of $SU(2)$ WZW model is generated by two copies of the $\widehat{SU}(2)$ current algebra at level k'

$$\begin{aligned}
\left[J_m^+, J_n^-\right] &= 2J_{m+n}^3 + k'm\delta_{m+n} \\
\left[J_m^3, J_n^\pm\right] &= \pm J_{m+n}^\pm \\
\left[J_m^3, J_n^3\right] &= \frac{k'}{2}m\delta_{m+n} ,
\end{aligned} \qquad (2.22)$$

and the Virasoro algebra given by the Sugawara form

$$L_n = \frac{1}{k'+2}\sum_{m=-\infty}^{\infty} :\delta_{ab}J_m^a J_{n-m}^b: . \qquad (2.23)$$

The representations of the $SU(2)$ WZW model are built from the familiar $SU(2)$ angular momentum representations D_j. A state is labeled as $|j, m, M\rangle$, with

$$
\begin{aligned}
L_0|j, m, M\rangle &= \left(\frac{j(j+1)}{k'+2} + M \right) |j, m, M\rangle \\
J_0^3|j, m, M\rangle &= m|j, m, M\rangle \ .
\end{aligned}
\tag{2.24}
$$

The zero modes of J^3 and $\bar{J}^3$ are related to translation along ψ direction in (2.21):

$$
-i\frac{\partial}{\partial\psi} = J_0^3 + \bar{J}_0^3 \ .
\tag{2.25}
$$

The possible values of j that may appear are restricted to $0 \leq j \leq k'/2$, in half-integer steps [44]. The complete Hilbert space of $SU(2)$ WZW model is therefore

$$
\mathcal{H}_{SU} = \oplus_{j=0,\frac{1}{2},\dots\frac{k'}{2}} \hat{D}_j \otimes \hat{D}_j \ .
\tag{2.26}
$$

2.4. Superstrings on $AdS_3 \times S^3 \times \mathcal{M}$

So far we have discussed the bosonic string theory. Our main interest is in the supersymmetric case, and in this subsection we will describe the supersymmetric extension of above discussion. For simplicity, we will limit our discussion to the $SL(2, R)$ model; the corresponding modifications for the $SU(2)$ model is straightforward. Further details on superstrings on group manifolds can be found in [45]. Superstrings on $AdS_3 \times S^3$ was also studied in [46], and the no-ghost theorem was proved in [47,48].

To extend the above results to the case of superstrings in RNS formalism, we need to introduce free worldsheet fermions χ^a which together with the total current K^a comprise the WZW supercurrent:

$$
C^a = \chi^a + \theta K^a \ ,
\tag{2.27}
$$

with θ a holomorphic Grassmann variable. The OPE's of K^a and χ^a are

$$
\begin{aligned}
K^a(z)K^b(w) &\sim \frac{k}{2}\frac{\eta^{ab}}{(z-w)^2} + \frac{i\epsilon^{ab}{}_c K^c(w)}{z-w} \\
K^a(z)\chi^b(w) &\sim \frac{i\epsilon^{ab}{}_c\chi^c(w)}{z-w} \\
\chi^a(z)\chi^b(w) &\sim \frac{k}{2}\frac{\eta^{ab}}{z-w} \ .
\end{aligned}
\tag{2.28}
$$

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This shows that K^a and χ^a do not form independent algebras. By subtracting the fermionic contribution to the total current, we obtain the bosonic current

$$k^a = K^a + \frac{i}{k}\epsilon^a{}_{bc}\chi^b\chi^c \, , \tag{2.29}$$

which have the OPE's

$$
\begin{aligned}
k^a(z)k^b(w) &\sim \frac{k+2}{2}\frac{\eta^{ab}}{(z-w)^2} + \frac{i\epsilon^{ab}{}_c k^c(w)}{z-w} \\
k^a(z)\chi^b(w) &\sim 0 \, .
\end{aligned}
\tag{2.30}
$$

Hence the level of the bosonic WZW model is shifted from k to $k+2$. Similarly, for the supersymmetric $SU(2)$ WZW model one introduces three fermions ζ^a which together with J^a form the supercurrent. The purely bosonic current j^a is defined analogous to (2.29), and the level of the bosonic part is shifted from k' to $k'-2$. The stress tensor and the Virasoro supercurrent are given by

$$
\begin{aligned}
T &= \frac{1}{k}(\eta_{ab}k^a k^b - \eta_{ab}\chi^a\partial\chi^b) + \frac{1}{k'}(\delta_{ab}j^a j^b - \delta_{ab}\zeta^a\partial\zeta^b) \\
G &= \frac{2}{k}\left(\eta_{ab}\chi^a k^b - \frac{i}{3k}\epsilon_{abc}\chi^a\chi^b\chi^c\right) + \frac{2}{k'}\left(\delta_{ab}\zeta^a j^b - \frac{i}{3k'}\epsilon_{abc}\zeta^a\zeta^b\zeta^c\right) \, .
\end{aligned}
\tag{2.31}
$$

Criticality of superstring theory on $AdS_3 \times S^3 \times \mathcal{M}$, where $\mathcal{M}$ is $K3$ or T^4, requires the central charge to satisfy

$$\frac{3(k+2)}{k} + \frac{3}{2} + \frac{3(k'-2)}{k'} + \frac{3}{2} = 9 \, , \tag{2.32}$$

which relates the levels of the current algebras

$$k = k' \, . \tag{2.33}$$

It is worthwhile to use variables commonly used when discussing AdS/CFT duality. In deriving the AdS_3/CFT$_2$ correspondence from D-branes, S-duality can be used to transform the D1-D5 system into an NS1-NS5 system. Taking the near horizon limit, the level of $SL(2,R)$ WZW model is identified with Q_5, the number of 5 branes (for details see [4], [49]). Hence the bosonic levels of the $SL(2,R)$ and $SU(2)$ WZW models are $Q_5 + 2$ and $Q_5 - 2$, respectively.

The supersymmetric generalization of spectral flow in $SL(2,R)$ WZW model was given in [50]. The spectral flow operation, given by the action of what was referred to as

the "twist field" in that work, not only induces transformation on the $\widehat{SL}(2,R)$ quantum numbers but also on the CFT describing the internal space. Physically, this coupling between the $SL(2,R)$ part and the internal CFT has its roots in the fact that in order for the spacetime theory to admit supersymmetry, one needs to pair χ^3 with a fermion from the internal CFT and then bosonize [51,52,53]. In the case of $AdS_3 \times S^3 \times \mathcal{M}$ the internal fermion is identified with ζ^3 and in the language of [50] every time the twist in $\widehat{SL}(2,R)$ is taken there is a corresponding twist in $\widehat{SU}(2)$.

Thinking of spectral flow as a twist is equivalent to the parafermion decomposition $SL(2,R) \simeq SL(2,R)/U(1) \times U(1)$ and $SU(2) \simeq SU(2)/U(1) \times U(1)$, in the following way. Introduce free bosons ϕ and ψ, normalized such that

$$\langle \phi(z)\phi(z') \rangle = \log(z - z') , \qquad \langle \psi(z)\psi(z') \rangle = -\log(z - z') . \tag{2.34}$$

In terms of which k_0^3 and j_0^3 can be expressed as

$$k^3(z) = -i\sqrt{\frac{k}{2}}\partial\phi , \qquad j^3(z) = -i\sqrt{\frac{k'}{2}}\partial\psi . \tag{2.35}$$

Throughout this discussion k and k' stand for the bosonic $SL(2,R)$ and $SU(2)$ levels, respectively. Then the bosonic $SL(2,R)$ primary field $\Phi_{ln\bar{n}}$ is decomposed into a field of $SL(2,R)/U(1)$ times a field in $U(1)$, where the $U(1)$ is generated by ϕ:

$$\Phi_{ln\bar{n}} = e^{in\sqrt{\frac{2}{k}}\phi + i\bar{n}\sqrt{\frac{2}{k}}\phi}\Phi_{ln\bar{n}}^{SL/U(1)} . \tag{2.36}$$

Similarly, a bosonic $SU(2)$ primary $\Psi_{jm\bar{m}}$ is written as

$$\Psi_{jm\bar{m}} = e^{im\sqrt{\frac{2}{k'}}\phi + i\bar{m}\sqrt{\frac{2}{k'}}\phi}\Psi_{jm\bar{m}}^{SU/U(1)} . \tag{2.37}$$

The fields $\Phi_{ln\bar{n}}^{SL/U(1)}$ are $\Psi_{jm\bar{m}}^{SU/U(1)}$ parafermions, with weight

$$\begin{aligned}
h(\Phi_{ln\bar{n}}^{SL/U(1)}) &= -\frac{l(l-1)}{k-2} + \frac{n^2}{k} , \\
h(\Psi_{jm\bar{m}}^{SU/U(1)}) &= \frac{j(j+1)}{k'+2} - \frac{m^2}{k'} ,
\end{aligned} \tag{2.38}$$

so that (2.36) and (2.37) have the expected weights. Note that under the shift $n \to n+wk/2$ and $m \to m + wk'/2$, the weights of the primary fields change to

$$\begin{aligned}
h(\Phi_{ln\bar{n}}) &\to -\frac{l(l-1)}{k-2} - nw - \frac{kw^2}{4} , \\
h(\Psi_{jm\bar{m}}) &\to \frac{j(j+1)}{k'+2} + mw + \frac{k'w^2}{4} .
\end{aligned} \tag{2.39}$$

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Spectral flow in the supersymmetric theory consists of the above shift in n, m, plus an additional contribution from the fermions [50], which gives

$$h(\Phi^w_{ln\bar{n}}) = -\frac{l(l-1)}{Q_5} - nw - \frac{Q_5 w^2}{4} ,$$
$$h(\Psi^w_{jm\bar{m}}) = \frac{j(j+1)}{Q_5} + mw + \frac{Q_5 w^2}{4} .$$

(2.40)

There is a similar relation on the anti-holomorphic side as well, with the same w. Note that the parafermion formalism also provides a convenient way of defining the vertex operators for states belonging to the spectral flowed representations [54,32,34]. The physical state condition is $(L_n - a\delta_{n,0})|\Omega\rangle = 0$ for $n \geq 0$, where $a = \frac{1}{2}$ in the NS sector and $a = 0$ in the R sector, as well as $G_r|\Omega\rangle = 0$ for $r \geq 0$. In addition, the analogue of GSO projection is the requirement of mutual locality with the supercharges that are constructed by bosonizing the worldsheet fermions [50].

3. Penrose limit of $AdS_3 \times S^3$ with NS background

In this section we explain the Penrose limit [8] of $AdS_3 \times S^3$ that results in the plane wave geometry [6,55].

The six dimensional plane wave is obtained from $AdS_3 \times S^3$ by expanding around a particular class of geodesics. These geodesics correspond to a particle near the center of AdS_3 and moving with very high angular momentum around a great circle of S^3. For this purpose, we begin with the spacetime metric

$$ds^2 = R^2(-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\phi^2 + \cos^2\theta\, d\psi^2 + d\theta^2 + \sin^2\theta\, d\varphi^2) + ds^2_{\mathcal{M}} \quad (3.1)$$

and introduce the coordinates

$$t = \mu x^+$$
$$\psi = \mu x^+ - \frac{x^-}{\mu R^2} .$$

(3.2)

Rescaling $\rho = r/R$, $\theta = y/R$, the metric is expanded around $\rho = \theta = 0$ by taking the limit $R \to \infty$. This results in the six dimensional plane wave

$$ds^2 = -2dx^+dx^- - \mu^2(r^2 + y^2)dx^+dx^+ + dr^2 + r^2d\phi^2 + dy^2 + y^2d\varphi^2 + ds^2_{\mathcal{M}} . \quad (3.3)$$

String spectrum in this background with NS three form field strength was found by quantizing the light cone action in [6,26,28]. For our purposes we will find it convenient to

take the light cone Hamiltonian as given in [27], adapted to the conventions of this paper and supersymmetrized,

$$H_{lc} = p^- = \mu(2 + q + \bar{q}) + \frac{N + \bar{N} + h^{\mathcal{M}} + \bar{h}^{\mathcal{M}} - 1}{p^+ \alpha'} . \qquad (3.4)$$

This applies to the NS-NS sector, and the last term needs to be appropriately changed for the R sector. The quantities appearing in this expression have the following physical interpretation. N is the total grade coming from the excitations along the pp-wave. $h^{\mathcal{M}}$ is the weight of the state coming from the CFT on $\mathcal{M}$. Finally, q is the net number of times the spacetime light cone energy raising and lowering operators have been applied to the ground state. The ground state in question may or may not be physical, i.e. we are referring to the ground state before the GSO projection. We have chosen the letter q to denote this number because as we shall see the physical meaning of this quantity is the same as the q we used in labelling the current algebra representations, see the remark below (2.17). There are corresponding contributions from the anti-holomorphic side to (3.4), subject to the constraint that the net momentum along the worldsheet vanishes,

$$N + h = \bar{N} + \bar{h} . \qquad (3.5)$$

The lightcone variables p^- and p^+ are related to observables measured in the global coordinates (3.1) by

$$\begin{aligned} p^- &= i\partial_{x^+} = \mu(E - J) \\ p^+ &= i\partial_{x^-} = \frac{J}{\mu R^2} . \end{aligned} \qquad (3.6)$$

E is the spacetime energy and J is the angular momentum around the ψ direction in S^3. Our choice of basis in labeling the $SU(2)$ representations (2.24) corresponds to diagonalizing the action of rotation in ψ, hence J is given by $J_0^3 + \bar{J}_0^3$.

The radius of AdS_3 and S^3 is related to Q_5 by $R^2 = \alpha' Q_5$, so the second equation in (3.6) is equivalent to

$$\mu p^+ \alpha' = \frac{J}{Q_5}. \qquad (3.7)$$

Hence the string spectrum in the NS-NS sector is

$$E - J = 2 + q + \bar{q} + \frac{Q_5}{J}(N + \bar{N} - 1) + \frac{Q_5}{J}(h^{\mathcal{M}} + \bar{h}^{\mathcal{M}}) , \qquad (3.8)$$

with the condition (3.5).

We make a few comments about the brane charges. Note that Q_1, the number of 1branes, actually never appears in any of the formulas[6]. But it should be kept in mind that Q_1 is being taken to infinity as well. As explained in [55], the plane wave limit can be described in terms of the brane charges by taking Q_1, $Q_5 \to \infty$, with fixed Q_1/Q_5. The scaling used to obtain the plane wave requires that finite energy excitations of the resulting geometry have Δ, $J \to \infty$ as $\sqrt{Q_1 Q_5}$, with finite $\Delta - J$. Since $Q_1 \propto Q_5$, this actually implies that Δ, $J \to \infty$ as $Q_5 \sim k$, the level of the current algebra. We could have seen this directly from the fact that J/R^2 is held fixed as the limit $R^2 \to \infty$ is taken, but then it would not be clear that Q_1 is scaled to infinity as well. Also note that in the case of $Q_5 = 1$, due to the aforementioned shift in the level of the bosonic WZW model the bosonic $SU(2)$ part has a negative level. This is in conflict with the well-known result that the $SU(2)$ level must be a non-negative integer. We will return to the issue of $Q_5 = 1$ later in section 7.

4. Nambu action near the origin of $AdS_3 \times S^3$

One of the things we want to understand is how the string spectrum on $AdS_3 \times S^3 \times \mathcal{M}$ reduces to (3.8) in the limit $Q_5, J \to \infty$. In order to answer this question we must first understand how (3.8) takes into account the spectral flow parameter w. In this section we explain the physical significance of spectral flow in the plane wave.

The plane wave limit described above is essentially a semi-classical expansion about $AdS_3 \times S^3$, combined with the unusual procedure of boosting to infinite (angular) momentum. Indeed, the large k limit in WZW models corresponds to the semi-classical limit, since the WZW action is proportional to k. Motivated by these concerns we will consider the Nambu action, upto quadratic order in the fields, of a string moving moving near $\rho \sim \theta \sim 0$ of $AdS_3 \times S^3$. When J is taken to be large, of order Q_5, the resulting action displays spectral asymmetry which is then related to spectral flow [32].

The Nambu action is given by

$$S = \frac{1}{2\pi\alpha'} \int d\tau d\sigma (\sqrt{|g|} - \epsilon_{ab} B_{\mu\nu} \partial_a X^\mu \partial_b X^\nu) \tag{4.1}$$

with g the induced metric and $B_{\mu\nu}$ the NS-NS two form. The non-zero components of the B field are

$$B_{t\phi} = \frac{1}{4}\alpha' Q_5 \cosh 2\rho \,, \quad B_{\psi\varphi} = \frac{1}{4}\alpha' Q_5 \cos 2\theta \,. \tag{4.2}$$

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[6] This is a feature of the NS1-NS5 description [49].

We will consider a string located at small values of ρ and θ, and moving along the ψ direction. Since we will be interested in states with fixed angular momentum around ψ, we take as our classical solution $\psi(\tau, \sigma) = \psi(\tau)$. This corresponds to a string collapsed to a point and rotating around a great circle[7]. The components of the induced metric $g_{ab} = G_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$ are, in the gauge $t = \tau$,

$$
\begin{aligned}
g_{00} &= \alpha' Q_5(-(1 + (X^a)^2) + \partial_0 X^a \partial_0 X^a + (1 - (Y^a)^2)(\partial_0 \psi)^2 + \partial_0 Y^a \partial_0 Y^a) \\
g_{01} &= \alpha' Q_5(\partial_0 X^a \partial_1 X^a + \partial_0 Y^a \partial_1 Y^a) \\
g_{11} &= \alpha' Q_5(\partial_1 X^a \partial_1 X^a + \partial_1 Y^a \partial_1 Y^a) \,,
\end{aligned}
\tag{4.3}
$$

where $X^1 + iX^2 = \rho e^{i\phi}$, and $Y^1 + iY^2 = \theta e^{i\varphi}$. The coupling to B field simplifies in this gauge to

$$
-\frac{Q_5}{2\pi} \int d\tau d\sigma (\rho^2 \partial_1 \phi - \theta^2 \partial_0 \psi \partial_1 \varphi) \,,
\tag{4.4}
$$

where we have used the fact that ψ has no dependence on σ,

$$
\int d\tau d\sigma \partial_0 \psi \partial_1 \varphi = \int d\tau d\sigma \partial_1 (\partial_0 \psi \varphi) = 0 \,.
\tag{4.5}
$$

The resulting action (4.1) shows that ψ is a cyclic coordinate. Hence, the conjugate momentum $J_0 = \frac{\partial L}{\partial(\partial_0 \psi)}$ is constant and it is advantageous to perform a Legendre transformation for ψ. The resulting Routhian,

$$
R(X^a, Y^a; J_0) = L - J_0 \partial_0 \psi \,,
\tag{4.6}
$$

is then the Lagrangian that describes the dynamics of X^a and Y^a, while treating J_0 as a constant of motion. The subscript 0 is added to J here to indicate that it is the angular momentum of the ground state, because we are discussing the point particle limit. Taking J_0 to be large, of order Q_5, the action for X^a and Y^a upto quadratic order in the fields is found to be

$$
\begin{aligned}
S = \frac{J_0}{2\pi} \int d^2\sigma \Big[1 &- \frac{1}{2}|\partial_0 \Theta|^2 + \frac{1}{2}\frac{1}{A^2}|(\partial_1 - iA)\Theta|^2 \\
&- \frac{1}{2}|\partial_0 \Phi|^2 + \frac{1}{2}\frac{1}{A^2}|(\partial_1 - iA)\Phi|^2 \Big] \,,
\end{aligned}
\tag{4.7}
$$

where $A = J_0/Q_5$, and $X^1 + iX^2 = \Phi$, $Y^1 + iY^2 = \Theta$. We see that Φ and Θ are two massless charged scalar fields on $R \times S^1$, coupled to a constant gauge field $A_a = A\delta_{a,1}$. As

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7 The importance of studying such solutions were pointed out in [56,57].

shown by Maldacena and Ooguri in [32], this implies that if A is not an integer, the states of Φ and Θ belong to the discrete representations with spectral flow number w equal to the integer part of A. Let us explain how this arises. The solution to the equation of motion that follows from (4.7) is

$$
\begin{aligned}
\Phi &= \sum_n \left(a_n^\dagger e^{i(n-A)(\tau/A+\sigma)} + b_n e^{-i(n-A)(\tau/A-\sigma)} \right) \frac{e^{iA\sigma}}{n-A} \\
\Theta &= \sum_n \left(c_n^\dagger e^{i(n-A)(\tau/A+\sigma)} + d_n e^{-i(n-A)(\tau/A-\sigma)} \right) \frac{e^{iA\sigma}}{n-A} \ .
\end{aligned}
\tag{4.8}
$$

Canonical quantization gives for the commutation relations

$$
\begin{aligned}
{[a_n, a_m^\dagger]} &\sim (n-A)\delta_{n,m} \ , &\quad [b_n, b_m^\dagger] &\sim (n-A)\delta_{n,m} \\
{[c_n, c_m^\dagger]} &\sim (n-A)\delta_{n,m} \ , &\quad [d_n, d_m^\dagger] &\sim (n-A)\delta_{n,m} \ .
\end{aligned}
\tag{4.9}
$$

Hence, for $n > A$, $a_n^\dagger$ is the creation operator while for $n < A$, a_n should be thought of as the creation operator. Similar comments apply to the other sets of operators. The holomorphic currents constructed from Φ and Θ are

$$
\begin{aligned}
K^+ &\sim -iQ_5 \sum_n a_n e^{-in(\tau/A+\sigma)} \\
K^- &\sim iQ_5 \sum_n a_n^\dagger e^{in(\tau/A+\sigma)} \\
J^+ &\sim -iQ_5 \sum_n c_n e^{-in(\tau/A+\sigma)} \\
J^- &\sim iQ_5 \sum_n c_n^\dagger e^{in(\tau/A+\sigma)} \ .
\end{aligned}
\tag{4.10}
$$

Each current may be mode expanded and using (4.9) the vacuum obeys

$$
\begin{aligned}
n > A : &\quad J_n^+ |0\rangle = 0 \ , \quad K_n^+ |0\rangle = 0 \ , \\
n > -A : &\quad J_n^- |0\rangle = 0 \ , \quad K_n^- |0\rangle = 0 \ .
\end{aligned}
\tag{4.11}
$$

Notice that this is different from the familiar highest weight conditions, which state that, for example, $K_{n>0}^+$ should annihilate the vacuum. The highest weight conditions can be restored by the transformation

$$
K_n^\pm = \tilde{K}_{n\mp w}^\pm \ , \qquad J_n^\pm = \tilde{J}_{n\mp w}^\pm \ ,
\tag{4.12}
$$

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with w an integer satisfying $w < A < w+1$. With respect to $\tilde{K}$ and $\tilde{J}$, the states created from $|0\rangle$ fill out the conventional highest weight representations. This shows that for J_0 not a multiple of Q_5, the states are in the discrete representations with spectral flow number equal to the integer part of J_0/Q_5.

On the other hand, when J_0/Q_5 is an integer, the $SL(2,R)$ part of the state is in the continuous representation with spectral flow number J_0/Q_5 [32].

The fact that spectral flow is necessary when J_0 is comparable to Q_5 should not be too surprising. In fact, the role of spectral flow is precisely to resolve the apparent conflict between the upper limit on $SL(2,R)$ spin of the discrete representations (2.14) and the freedom to have arbitrarily high angular momentum on S^3. More generally, for spacetimes of the form $AdS_3 \times \mathcal{N}$, the analysis of [32] shows that the amount of spectral flow is determined by ratio of the conformal weight h coming from the operator of the $\mathcal{N}$ CFT to the $SL(2,R)$ level k,

$$ w < \sqrt{\frac{4h}{k}} < w+1 \,. \tag{4.13} $$

For the case at hand, we see that $4h$ can be approximated as J_0^2/k and using $k \sim Q_5$ this reproduces what we found above.

What is surprising, however, is that (4.7) and the arguments that follow it imply that spectral flow should also be taken in the $SU(2)$ theory, with the same amount as the $SL(2,R)$ part. To be sure, this is not to suggest that the Hilbert space of $SU(2)$ WZW model needs to be enlarged to include spectral flowed representations, similar to what was done in the case of $SL(2,R)$ model. Whereas the $\widehat{SL}(2,R)$ representations generated by spectral flow are new and distinct from the conventional representations, this is not true in the case of $\widehat{SU}(2)$ representations. But as reviewed in section 2 supersymmetry requires that spectral flow is taken in both WZW models. Due to the high number of supersymmetries possible on this background[8] it is not unreasonable to think that this peculiar feature of the supersymmetric theory manifests itself in the purely bosonic analysis presented here. Additionally, note that the action of spectral flow on the angular momentum generator,

$$ J_0^3 \rightarrow J_0^3 + \frac{wk}{2}, \tag{4.14} $$

---

[8] String theory on $AdS_3 \times \mathcal{N}$ generically has $N = 2$ spacetime supersymmetry if $\mathcal{N}$ has an affine $U(1)$ symmetry and the coset $\mathcal{N}/U(1)$ admits a $N = 2$ superconformal algebra. In the case $\mathcal{N} = S^3 \times \mathcal{M}$, supersymmetry is enhanced to $N = 4$ [52,53,58].

has the right form to be useful in keeping track of states with $J \sim k$ while k is taken to infinity. This feature makes it worthwhile to introduce spectral flow in the $SU(2)$ WZW model[9]. In the next section, we will use this idea to obtain the large J spectrum of superstrings on $AdS_3 \times S^3$.

5. The plane wave spectrum

We now turn to explaining how the plane wave spectrum arises from the exact $AdS_3 \times S^3$ results. The discussion will be limited to the NS sector, as the R sector can be obtained by similar methods, with the additional use of the spin fields.

5.1. Short strings

We start with the discrete $w = 0$ states, the holomorphic side of which is labeled by the quantum numbers

$$|\ell, n, N\rangle \otimes |j, m, M\rangle \otimes |h^{\mathcal{M}}\rangle . \tag{5.1}$$

The notation in labeling the $\widehat{SL}(2, R) \times \widehat{SU}(2)$ part of the state is the same as what was used in section 2, and $h^{\mathcal{M}}$ is the conformal weight coming from the CFT on $\mathcal{M}$. In order for (5.1) to be physical, it must satisfy

$$-\frac{\ell(\ell - 1)}{Q_5} + \frac{j(j + 1)}{Q_5} + N + M + h^{\mathcal{M}} = \frac{1}{2} . \tag{5.2}$$

Let us look for the ground state within a given j sector. First, we note that the GSO projection [50] requires the lowest excitation number to be one half, so from (5.2) we find $\ell = j + 1$. Next, we see that the lowest value of energy (for fixed j) is obtained if this one half unit of excitation comes from the action of $\zeta^+_{-1/2}$ or $\chi^-_{-\frac{1}{2}}$. In the first case, the ground state is

$$|J/2, J/2\rangle \otimes \zeta^+_{-\frac{1}{2}} |J/2 - 1, J/2 - 1\rangle \otimes |0\rangle , \tag{5.3}$$

and in the second,

$$\chi^-_{-\frac{1}{2}} |J/2 + 1, J/2 + 1\rangle \otimes |J/2, J/2\rangle \otimes |0\rangle . \tag{5.4}$$

Combining with an identical state in the anti-holomorphic side, we see that there is a total of four states that carry angular momentum J and energy $E = J$, i.e. the light cone vacuum.

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[9] See [59] for an interesting application of spectral flow in the $SU(2)$ WZW model.

We will not discuss the Ramond sector in detail, but in order to complete the discussion of light cone ground states we briefly mention how many are found in the Ramond sector. The number of light cone ground states coming from the Ramond sector depends on whether $\mathcal{M}$ is T^4 or $K3$. For T^4, there are two ground states in the R sector, and one can construct the usual NS-NS, NS-R, R-NS, R-R sectors to find a total of 16 ground states [60]. When the internal manifold is $K3$, for the purposes of counting ground states we can think of T^4/Z_2 instead. Then, as explained in [28], the ground states in the NS-R and R-NS sectors are projected out, and the 16 twisted sectors each give a ground state in the R-R sector. Thus there are 24 ground states in all, as expected.

The excited states of $w = 0$ representations are obtained from the lowest weight of $SL(2,R)$ and the highest weight of $SU(2)$ by the action of negatively moded generators. Physical states do not carry excitations along the time direction. For example, in the $SL(2,R)$ Hilbert space those states satisfying the Virasoro conditions can be written

$$\prod_{r=1/2}^{\infty} (\chi_{-r}^+)^{N_r^+}(\chi_{-r}^-)^{N_r^-} \prod_{n=0}^{\infty} (k_{-n}^+)^{N_n^+}(k_{-n}^-)^{N_n^-} |\ell, n = \ell\rangle \,, \qquad (5.5)$$

which has the grade

$$N = \sum_n n(N_n^+ + N_n^-) + \sum_r r(N_r^+ + N_r^-) \qquad (5.6)$$

and $n = \ell + q_{SL}$, with

$$q_{SL} = \sum_n (N_n^+ - N_n^-) + \sum_r (N_r^+ - N_r^-) \,. \qquad (5.7)$$

Similar relations hold for the $SU(2)$ part. Now (5.2) is used to solve for ℓ, which then gives for the energy

$$E = 1 + q_{SL} + \bar{q}_{SL} + \sqrt{(2j+1)^2 + 2Q_5(N + \bar{N} + M + \bar{M} + h^{\mathcal{M}} + \bar{h}^{\mathcal{M}} - 1)} \,, \qquad (5.8)$$

with j related to J by $J = 2j - q_{SU} - \bar{q}_{SU}$. Now we take the "Penrose limit" $Q_5, J \to \infty$ with J/Q_5 fixed, and expanding to terms of order one we find

$$E - J = 2 + q_{SL} + \bar{q}_{SL} + q_{SU} + \bar{q}_{SU} + \frac{Q_5}{J}(N + \bar{N} + M + \bar{M} + h^{\mathcal{M}} + \bar{h}^{\mathcal{M}} - 1) \,. \qquad (5.9)$$

Note that the vacuum states considered above corresponds to sum of the q's totalling -2 and total grade equal to 1. That the lowest energy state surviving the GSO projection in

the NS sector has a half unit of excitation is similar to what happens in flat space. The difference in this case is that the various raising operators have different charges under E and J. Note also that the $w = 0$ continuous representations are projected out from the physical spectrum, since for those representations it is impossible to satisfy the physical state condition unless $N = 0$. Hence the spectrum is free of tachyons.

Having understood the $w = 0$ states, we now turn to the spectral flowed states. Consider a state in the spectral flowed representation of $\widehat{SL}(2, R) \times \widehat{SU}(2)$, tensored with an operator on $\mathcal{M}$,

$$|w, \tilde{\ell}, \tilde{n}, N\rangle \otimes |w, \tilde{j}, \tilde{m}, M\rangle \otimes |h^{\mathcal{M}}\rangle . \tag{5.10}$$

There is a similar state on the anti-holomorphic side. Using (2.40), the physical state condition determines $\tilde{\ell}$ to be

$$2\tilde{\ell} = 1 - Q_5 w + \sqrt{(2\tilde{j} + Q_5 w + 1)^2 + 2Q_5 \left(N + \bar{N} + M + \bar{M} - 2w + h^{\mathcal{M}} + \bar{h}^{\mathcal{M}} - 1\right)} , \tag{5.11}$$

where we have used the second equation in (2.40) for the weight of the $SU(2)$ state. In this relation N and M are the grades measured by L_0, not $\tilde{L}_0$, of the $SL(2, R)$ and $SU(2)$ model respectively. Now we can use $J = 2\tilde{j} + Q_5 w - q_{SU} - \bar{q}_{SU}$ to substitute for $\tilde{j}$ in the expression above, and the energy is given by

$$E = 2\tilde{\ell} + Q_5 w + q_{SL} + \bar{q}_{SL} . \tag{5.12}$$

This result is an exact formula for the energy of a string state in $AdS_3 \times S^3 \times \mathcal{M}$ with angular momentum J around S^3.

Taking the limit $Q_5, J \to \infty$ and expanding to terms of order one,

$$E - J = 2 + q_{SL} + \bar{q}_{SL} + q_{SU} + \bar{q}_{SU} + \frac{Q_5}{J}(N + \bar{N} + M + \bar{M} - 2w - 1) + \frac{Q_5}{J}(h^{\mathcal{M}} + \bar{h}^{\mathcal{M}}) . \tag{5.13}$$

The states with $E = J$ again have the form (5.3) or (5.4), but now there is a slight difference due to spectral flow. For example, in the spectral flowed analogue of (5.3), fermionic generator is given by $\tilde{\zeta}^+_{-\frac{1}{2}}$, which has $M = \frac{1}{2} + w$ after taking into account the shift in moding from spectral flow. This serves to cancel the extra term in (5.13) compared to (5.9). As found in [50], the pattern of chiral states is relatively simple. Once the $w = 0$ chiral states are identified, spectral flow generates the chiral states with higher R-charge. In general a state similar to (5.5) in a spectral flowed representation has $\tilde{N}$ and $\tilde{q}_{SL}$ defined

in the same manner as (5.6) and (5.7), respectively. They are related to what appear above as

$$N = \tilde{N} - w\tilde{q}_{SL} \ ,$$

$$q_{SL} = \tilde{q}_{SL} \ . \tag{5.14}$$

In the semiclassical discussion of the previous section we saw that the amount of spectral flow necessary is determined by the ratio J_0/Q_5, where J_0 is the angular momentum of the ground state, i.e. a state in the zero grade of a $\widehat{SU}(2)$ representation. In the fully quantum treatment, w is determined through the inequality $\frac{1}{2} < \tilde{\ell} < \frac{Q_5+1}{2}$, which becomes

$$w^2 < \frac{(2\tilde{j} + Q_5 w + 1)^2}{Q_5^2} + \frac{2}{Q_5}(N + \bar{N} + M + \bar{M} - 2w + h^{\mathcal{M}} + \bar{h}^{\mathcal{M}} - 1) < (w+1)^2 \ . \tag{5.15}$$

It should be remembered that N and M also depend on w, through (5.14) and an analogous relation for M. In (5.15) we can think of $\tilde{j} + Q_5 w/2$ as the highest weight of the $SU(2)$ representation from which the current algebra representation is constructed,

$$J_0 = 2\tilde{j} + Q_5 w \ , \tag{5.16}$$

and (5.15) reproduces the semiclassical result found previously.

5.2. Long strings and the "missing" chiral primaries

Let us now discuss what happens when the inequality in (5.15) is saturated, which in the semiclassical approximation corresponds to J_0/Q_5 becoming an integer. In this case we know from [32] that the state belongs to a continuous representation of $\widehat{SL}(2, R)$ with spectral flow number $w = J_0/Q_5$, i.e. it is a long string in AdS_3. Morever, the energy of the solution changes smoothly in the transition from a short string to a long string (and vice versa). The continuous representations do not have highest or lowest weights and for this reason the spectral flowed states are labelled by the eigenvalues of $\tilde{L}_0$. The plane wave spectrum of the long strings is therefore

$$E - J = 2 + \frac{Q_5}{J}(\tilde{N} + \tilde{\tilde{N}} + \tilde{M} + \tilde{\tilde{M}} - 2w - 1) + \frac{Q_5}{J}(h^{\mathcal{M}} + \bar{h}^{\mathcal{M}}) \ . \tag{5.17}$$

Sometimes it is possible for a long string to have zero light cone energy despite the fact that it is massive. If $|0, w\rangle$ denotes a state with $E = J$ then $k_w^+|0, w\rangle$ continues to have zero light cone energy because k_w^+'s contribution to (5.17), proportional to $\tilde{N}$, vanishes. The physical mechanism responsible for this phenomenon is the same as in AdS_3. Namely,

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the coupling to the NS three form cancels the gravitational attraction. In the context of plane waves supported by NS field strengths it has already been observed that there are additional zero modes in the spectrum [6,29,59], which can be understood as the statement that states with special values of p^+ — integer multiples of $1/\mu\alpha'$ — do not feel the confining potential of the plane wave.

It is interesting to note that simplifying $AdS_3 \times S^3$ to the plane wave makes more apparent the presence of long strings in the spectrum. As we have just stated, some of the long strings correspond to chiral primaries in the dual CFT. It has been appreciated for a while now that there is a mismatch of chiral primaries in the AdS_3/CFT_2 correspondence when considering the AdS_3 with a purely NS background due to the fact that AdS_3 with vanishing R-R fields corresponds to a "singular" CFT [42,50]. The mismatch arises when one tries to compare the spectrum of chiral primary operators in the CFT to the spectrum of chiral string states based on the discrete representations of $\widehat{SL}(2,R)$. It was suggested in [42] that the chiral primaries that disappear when all the R-R fields are set to zero might be found among the continuum. We find explicitly that indeed there are chiral primaries belonging to the continuous representations.

6. The decomposition of the Hilbert space in the Penrose limit

We started with a unitary spectrum of string states in $AdS_3 \times S^3 \times \mathcal{M}$. This spectrum is obtained from the Hilbert space of the $SL(2,R)$ WZW model, tensored with the Hilbert spaces of the $SU(2)$ model and CFT on $\mathcal{M}$, and imposing the Virasoro constraints. In obtaining the results of previous section we have restricted our focus to a particular subsector of this physical Hilbert space. We now address the question of what happens to the remaining states in the Hilbert space. We find that the ratios J/Q_5, J^2/Q_5 determine where the state ends up.

As we take the limit $R \to \infty$, we expect that some of the states become strings in flat space, some become strings in the plane wave, and the rest with divergent $E - J$. The spectra in flat space and plane wave should form independent, unitary Hilbert spaces. Presumably, this means that the states with divergent $E - J$ should also, but with a different description. An example of such states would be those that have high angular momentum along a different circle on S^3. These states would be related to what we considered above by a global rotation on the sphere.

We have found the states on the plane wave. Which states correspond to strings in flat space? In any dimension, flat space is obtained from plane wave when [6]

$$\mu\alpha'p^+ << 1 \ . \tag{6.1}$$

But in our case, $\mu\alpha'p^+ = J/Q_5$, and we know that the integer part of J/Q_5 is related to the spectral flow parameter w in the large J, Q_5 limit. Thus we conclude that the flat space spectrum comes from the unflowed short strings in the original AdS_3 theory. We can indeed check that for $J/Q_5 \rightarrow 0$, J^2/Q_5 finite, the physical state condition for $w = 0$ short strings (5.2) reproduces the mass formula of superstrings in six flat dimensions times $\mathcal{M}$, because the terms in L_0 that involve the quadratic Casimirs become p^2 as the space becomes flat [44].

It is important to note that even though we have just identified the flat space spectrum as arising from the $w = 0$ sector of the original theory, this does not mean that none of the $w = 0$ short strings remain in the plane wave. Some of the states can still carry $J \sim Q_5$, and as the limit $Q_5 \rightarrow \infty$ is taken we find the result (5.9). However, the $w = 0$ plane wave states are generally farther from chiral than the spectral flowed states.

If $J^2/Q_5 \rightarrow 0$, then (5.9) tells us that the string modes have energy that diverges as $\sqrt{Q_5}$. Note, however, that even in this case the supergravity modes (i.e. states at grade $1/2$ for both the right and left movers) remain, and they fall into the global $SL(2, R) \times SU(2)$ multiplets.

7. When the radius is small

An extremely interesting question one would like to address is what we can learn about string theory on $AdS \times S$ from string theory on plane waves. In the case of $AdS_3 \times S^3$ and its plane wave limit, we have a good understanding of both string theories, and we now turn to this question.

But first, we'd like to stress a small point, which is that a priori there are two distinct notions of "high curvature" one needs to keep in mind. When one speaks of a highly curved plane wave, that actually means

$$\mu\alpha'p^+ >> 1 \ . \tag{7.1}$$

In this case the string spectrum consists of highly spectral flowed states. We see from (5.13) that this means the low lying string modes become almost degenerate. This is similar to what happens in the $AdS_5 \times S^5$ plane wave.

Despite being "highly curved", the highly curved plane wave still involves taking the radii of $AdS \times S$ to infinity. Hence the GS superstring in highly curved plane waves is still amenable to quantization. The second, and more interesting, notion of "high curvature" is obtained by dropping the $R^2 \to \infty$ condition. Then clearly the geometry cannot be thought of as a plane wave. Since it is only after the Penrose limit is taken that the GS string can be solved, presently known results about the plane wave of $AdS \times S$ are not expected to remain valid in the case of small radius.

However, there have been some reasons to think that the plane wave spectrum (3.8) might continue to correctly describe the large J spectrum even outside the strict $Q_5 \to \infty$ limit. Authors of [55] studied various aspects of string theory on the plane wave (3.3) from the point of view of the dual $(\mathcal{M})^{Q_1 Q_5}/S_{Q_1 Q_5}$ CFT. One of the more interesting things they found in that work was that after extrapolating (3.8) to $Q_5 = 1$, the result surprisingly agrees with the spectrum predicted by the dual CFT at the orbifold point[10]. Since the CFT spectrum is believed to be reliable for arbitrary Q_5, whereas the string spectrum was found under the assumption that Q_5 is taken to infinity, this hints that perhaps (3.8) is true even when the spacetime geometry does not correspond to a plane wave. There have also been some work along this line for the $AdS_5 \times S^5$ plane wave [37,38], but with some differences, which we will discuss in the last section.

We can answer this question directly for the $AdS_3 \times S^3$ case since we worked out the string spectrum that is valid for all values of Q_5. Our results apply equally to small Q_5, when we should think of the geometry as $AdS_3 \times S^3 \times \mathcal{M}$ with the first two factors being highly curved. Thus, we can take (5.11), (5.12) and expanding for arbitrary fixed Q_5, large w, we find that, in fact, the large J spectrum is again given by (5.13). We conclude that the plane wave spectrum is actually the large J spectrum of strings on $AdS_3 \times S^3 \times \mathcal{M}$, for arbitrary values of the radius.

Actually, there are two special cases where the worldsheet description we have given so far could break down. These special cases occur for $Q_5 = 1$ or 2, whereby due to the shift in the level of the bosonic WZW models the $SU(2)$ model acquires a negative or zero level. However, the problem is not serious for the $Q_5 = 2$ case as we can understand it to mean that only the fermionic fields are present on the worldsheet for the S^3 part of

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[10] In fact, the NS $Q_5 = 1$ is the only case where a perfect agreement was found. Matching of the spectra in general requires g_s^2 corrections and on the CFT side involves moving away from the orbifold point in the moduli space.

the target space. The $Q_5 = 1$ case truly presents us with a difficulty since it is not clear how to make sense of the $SU(2)$ WZW model at level -1 as a physical theory. It is not known at present how to describe the $Q_5 = 1$ model, but arguments were presented in [42] to suggest that it is a sensible (albeit very special) system. We'd like to argue that the result (5.13) is valid even for the $Q_5 = 1$ case even though our starting point was not suited to describe it. For one, it would be rather unusual for the expression (5.13) to be true for $Q_5 = 2, 3, \ldots \infty$ and not be true for $Q_5 = 1$ when nothing special happens as we try to set $Q_5 = 1$. More importantly, the dual CFT is well defined at $Q_5 = 1$ and its prediction for the string spectrum [55] matches perfectly with (5.13). Perhaps $Q_5 = 1$ actually represents the zero radius limit of AdS_3, thus providing the reason behind perfect agreement with the symmetric orbifold. The orbifold point of the CFT corresponds to the free theory (analogous to setting $g_{YM} = 0$ in AdS_5/CFT_4), whose dual string theory would apparently be formulated on zero radius AdS_3. We will return to this issue in the Discussion.

8. Summary and Discussion

The two main objectives of this paper have been

(a) To provide a CFT description of strings in a plane wave background, giving the necessary framework for a detailed study of BMN correspondence using the powerful tools of CFT.

(b) To investigate the relationship between string theory on $AdS \times S$ and string theory on plane waves, using the solvable $AdS_3 \times S^3$ case as a model.

We offer some comments on each of these issues.

It is worth emphasizing that we are now positioned to take advantage of the CFT techniques to study string interactions in the $AdS_3 \times S^3$ plane wave. This is in stark constrast to the much studied case of the ten dimensional plane wave that arises from $AdS_5 \times S^5$, where the RNS description of strings is lacking and interactions can only be studied using string field theory. In fact, correlation functions in AdS_3 have already been calculated [34], so together with the correlation functions of $SU(2)$ WZW model it should be possible to obtain scattering amplitudes in the plane wave by appropriately taking the large J, Q_5 limit. This should prove to be an useful area for study.

In regards to the AdS_3 correlation functions, we show in Appendix A that the spectral flow number violation rule found in [34] can be understood as the conservation of angular momentum in the plane wave.

Additionally, one expects that the map between the CFT operators and plane wave string states is easier to establish than the ten dimensional case, owing to the fact that the AdS_3/CFT_2 duality is highly constrained by the infinite dimensional conformal symmetry. Thus, it becomes a technically simpler problem to study the BMN correspondence in situations where many interacting string modes are involved.

The other main point of this paper is that we have actually compared string theory on $AdS_3 \times S^3$ to string theory on the plane wave. We have found that the plane wave spectrum, which one might have thought to be the result of some simplification of the $AdS_3 \times S^3$ spectrum that occurs in the limit $Q_5, J \to \infty$, actually is the result of $J \to \infty$ only.

Recently it has been conjectured by Frolov and Tseytlin [35,36,39] that the semi-classical formula for the energy of strings carrying spins in multiple directions in $AdS_5 \times S^5$ continues to hold true at small values of the radius, provided that the spins take very large values[11]. Based on the findings of this paper, we feel strongly that their conjecture is true. Furthermore, if the relationship between string theory on $AdS_3 \times S^3$ and its plane wave limit applies to other $AdS \times S$ spaces, it suggests that the string spectrum on the plane wave limit of $AdS_5 \times S^5$ [7,6] is in reality the large J string spectrum on $AdS_5 \times S^5$.

Before leaving the subject of the Frolov-Tseytlin solution, let us note a curious fact. Frolov and Tseytlin found that the solution carrying two non-zero equal spins in S^5 has the energy

$$E = \sqrt{(2J)^2 + \frac{R^4}{\alpha'^2}} \; . \tag{8.1}$$

This bears striking resemblance to the energy of a low-lying short string state in $AdS_3 \times S^3$ with the single spin

$$E \sim \sqrt{J^2 + c\frac{R^2}{\alpha'}} \; , \tag{8.2}$$

where c is a number of order 1. Other than the difference in the power of R^2/α', which could be explained by the fact that the role of N in AdS_5/CFT_4 is played by both $Q_1 Q_5$ and $\sqrt{Q_1 Q_5}$ in AdS_3/CFT_2 [55], the two expressions are almost identical. It should be kept in mind that (8.1) is a classical result whereas (8.2) is a quantum one. It is not clear if Frolov-Tseytlin solution has an interpretation as giving arise to a simpler spacetime geometry in a manner similar to BMN. However, as we have seen, strings with large J in $AdS_3 \times S^3$ have a simple description even though it is only after the radius is taken to be

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[11] See [61] for a discussion regarding the supersymmetry of the spinning strings.

large as well that they can be viewed as moving in the plane wave. At any rate, it would be extremely interesting to understand why these two expressions are so similar. Perhaps studying strings on $AdS_3 \times S^3 \times S^3 \times S^1$ [62], which makes multi-spin solutions possible, along the lines of this work will shed light on this issue[12].

Another topic of interest has been pursued in [37,38,63,64,65,66,67] involving strings in the critical tension limit and the possibility of defining string theory in the zero radius limit of AdS. The hope is to take the $\lambda \to 0, N \to \infty$ limit of AdS/CFT at its face value and establish a duality between string theory in the zero radius AdS and a free field theory. We should mention from the start, however, that the approach has been to send R^2/α' to zero in the classical Hamiltonian and then quantize the resulting (simpler) theory. This by no means assures us that we will find the same results when we take the same limit in the quantum theory. Another point to keep in mind is that when the radius of the spacetime is comparable to the string scale, it is not clear whether one can even assign a definitive value to the radius.

Now we focus on the $AdS_3 \times S^3$ example and try to address this issue. Strictly speaking, one must set $Q_5 = 0$ to study the zero radius AdS_3. In this case we do not know how to make sense of the worldsheet theory. However, as stated above we do not believe that one should insist on being able to set R^2/α' exactly to zero in the quantum treatment. For the time being, we will be content with considering $R^2 \sim \alpha'$, which is still a nontrivial case. It is perhaps useful to recast the large J expansion of the exact energy formula found in section 5 using the radius of curvature in string units (we ignore the internal space $\mathcal{M}$ for this discussion, whose contribution is suppressed anyway):

$$E - J = 2 + q_{SL} + \bar{q}_{SL} + q_{SU} + \bar{q}_{SU} + \frac{R^2}{\alpha' J}(N + \bar{N} + M + \bar{M} - 2w - 1) . \qquad (8.3)$$

We should note that the last terms in parantheses is what gives (8.3) its stringy nature. If for some reason (such as simply taking the "tensionless" limit $R^2/\alpha' = 0$ while continuing to trust (8.3)) they were absent, what remains would resemble a field theory spectrum. It might seem at first that the last terms would be negligible for large J, finite R^2/α'. But in fact this is not the case, because the excited string modes generically have grade of order

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[12] The author would like to thank A. Adams for this point.

$\alpha' J/R^2$ due to spectral flow. The only way in which the last terms in (8.3) disappear is in strict $R^2/\alpha' J = 0$ case[13]. When that happens the spectrum can be schematically written

$$H_{lc} \sim \sum_{all\ modes} a^\dagger a \,, \qquad (8.4)$$

which looks like a free field theory[14]. This suggests that the theory with $R^2/\alpha' = 0$ (whatever its proper description might be) is not continuously connected to the $R^2 \sim \alpha'$ cases at finite J.

In a related topic, authors of [37], [38] found evidence that the string spectrum on the plane wave limit of $AdS_5 \times S^5$ may be extrapolated down to finite J after setting g_s to zero, which has the effect of reducing the spectrum to the form (8.4). The agreement with the SYM prediction (which was done in [38] for conformal weights upto 10) as well as considerations of this paper lend support to the claim that in fact the entire string spectrum on $AdS_5 \times S^5$ reduces to (8.4) at $g_s = 0$.

There have also been some work on computing R^{-2} corrections to the plane wave spectrum as a way of approximating the $AdS \times S$ spectrum [68,69,27]. The results of this paper might be useful as a guide in checking higher order calculations. It is important to note, however, that in computing corrections to the plane wave one does not have the freedom to choose R^2 and J independently. The advantage we had in the $SL(2,R) \times SU(2)$ model was being able to vary Q_5 and J in an independent manner.

In conclusion, strings in $AdS_3 \times S^3$ and its plane wave or its large J limit seem to be very useful models to study and it is hoped that they will lead to a better understanding of the more complicated plane wave/CFT and AdS/CFT dualities.

Acknowledgements

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[13] Note that the combination $R^2/\alpha' J$ is the square root of the coupling constant λ' identified in the BMN limit of $AdS_5 \times S^5$ [9,11].

[14] However, not all information about string excitation numbers seems to be lost since the $L_0 = \bar{L}_0$ constraint still needs to be imposed.

Appendix A. The spectral flow number violation rule

In [34] it was found that the N-point function of vertex operators with spectral flow numbers w_i, viewed as describing the interaction of $i = 2, \ldots, N$ incoming strings and $i = 1$ outgoing string, vanishes unless[15]

$$w_1 \leq \sum_{i=2}^{N} w_i + N - 2 \ . \tag{A.1}$$

This result was derived using representation theory of $\widehat{SL}(2, R)$ algebra, irrespective of what the spacetime consists of besides AdS_3, and does not rely on any particular physical picture.

We now show that when considering the plane wave limit of $AdS_3 \times S^3 \times \mathcal{M}$, (A.1) can be understood as enforcing the conservation of J. In order to find a non-zero correlation function the J_i must satisfy

$$J_1 = \sum_{i=2}^{N} J_i \ . \tag{A.2}$$

We now divide both sides of this equation by Q_5 and identify w_i as the integer part of J_i/Q_5 (see the footnote below and also note that we are in the $J, Q_5 \to \infty$ regime). On the RHS, there will be $N - 1$ terms, each of the form $w_i + \Delta_i$ where $0 \leq \Delta_i < 1$. The sum of Δ_i's will therefore be less than $N - 1$. Hence the spectral flow numbers will satisfy (A.1).

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[15] The discrete states are taken to be in the ground states of their representations, i.e. $\tilde{n}_i = \tilde{\ell}_i$.

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